Lent 2019
v2

INTRODUCTION TO APPROXIMATE GROUPS—EXAMPLES 1

Please email comments and corrections to mcht2@cam.ac.uk.

1. Let G be an abelian group. Given a finite subset $A \subset G$, define $E(A) = |\{(a,b,c,d) \in A \times A \times A \times A : a+b=c+d\}|$.
   (a) Show that $2|A|^2 - |A| \leq E(A) \leq |A|^3$, and give examples attaining each of these bounds.
   (b) Show that for an arbitrary finite set $A \subset G$ the doubling constant is at least $|A|^3/E(A)$.

2. For each $k, n \in \mathbb{N}$ with $n \geq k$, let $A_{k,n}$ be a subset of $\{1, \ldots, n\}$ of size k chosen uniformly at random.
   (a) Defining $E(A_{k,n})$ as in Question 1, show that for each k we have $\mathbb{E}[\,E(A_{k,n})\,] \leq \frac{1}{n}(k^4+2k^3)+2k^2+k$.
   (b) Hence or otherwise show that for each k we have
   $$\liminf_{n\to\infty} \mathbb{E}[|A_{k,n}+A_{k,n}|] \geq \frac{k^2}{2+\frac{1}{k}}.$$
   as $n \to \infty$.
   (c) Define $D(k) = \max\{|A+A| : A \subset \mathbb{Z}, |A| = k\}$, and deduce that
   $$\frac{\liminf_{n\to\infty} \mathbb{E}[|A_{k,n}+A_{k,n}|]}{D(k)} \to 1$$
   as $k \to \infty$.

3. Let A be a finite subset of an arbitrary group.
   (a) Show that if $|A^2| < 2|A|$ then $A^{-1}A = AA^{-1}$.
   (b) Show that if $|AA^{-1}A| < 2|A|$ then $H = AA^{-1}$ is a subgroup such that $A \subset Ha$ for every $a \in A$.

4. Let A be a finite subset of an arbitrary group.
   (a) Show that if $|A^2| \leq K|A|$ then $|A^{-1}A| \leq K^2|A|$ and $|AA^{-1}| \leq K^2|A|$.
   (b) Conversely, does $|A^{-1}A| \leq K|A|$ or $|AA^{-1}| \leq K|A|$ imply a bound of the form $|A^2| \leq f(K)|A|$?
   (c) For which values of $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$ does the bound $|A^{\epsilon_1}A^{\epsilon_2}A^{\epsilon_3}| \leq K|A|$ imply a bound of the form $|A^3| \leq f_{\epsilon_1,\epsilon_2,\epsilon_3}(K)|A|$?

5. Let A, B be finite subsets of an abelian group G. Show that if $|A+B| \leq K|A|$ then $|mB - nB| \leq K^{m+n}|A|$ for all non-negative integers m, n. *Hint: Start by letting U be a non-empty subset of A that minimises the ratio $|U+B|/|U|$, writing $R = |U+B|/|U|$, and showing that $|U+mB| \leq R^m|U|$ for every $n \geq 0$.*

6. Let A be a finite subset of an abelian group and suppose that $|A+A| \leq K|A|$. Show that $2A - 2A$ is a K^{16}-approximate group. Can you also show that $A - A$ is an approximate group?

7. [*May be submitted for marking.*] Let A be a subset of an arbitrary group G and suppose that $|A^2| \leq K|A|$. Show that there is an $O(K^{48})$-approximate group B satisfying $|B| \leq 7K^2|A|$ and sets $X, Y \subset G$ of size at most $7K^2$ such that $A \subset XB \cap BY$.

8. Let G be a group, and let $A \subset G$ be a finite symmetric subset containing the identity. Show that if $|A^{2n+1}| \leq K|A^n|$ for some $n \in \mathbb{N}$ then there is an $f(K)$-approximate group B with $A^n \subset B$ and $|B| \leq K|A^n|$.

9. Let G be a group.
   (a) Show that if A is a K-approximate subgroup of G and B is an L-approximate subgroup then $A^m \cap B^n$ is a $K^{2m-1}L^{2n-1}$-approximate subgroup for every $m, n \geq 2$.

(b) Show that if A and B are finite symmetric subsets of G satisfying $|A^3| \leq K|A|$ and $|B^3| \leq L|B|$ then $|(A^m \cap B^n)^3| \leq (K^m L^n)^C |A^m \cap B^n|$ for every $m, n \geq 2$, with C an absolute constant that you should specify.

(c) Show that there exists $K \geq 1$ such that the following holds: if H is an arbitrary group and $B \subset H$ is an arbitrary finite subset then there exists a group G with $H < G$ and a subset $A \subset G$ with $|A^3| \leq K|A|$ such that $A \cap H = B$. Deduce that part (b) does not necessarily hold if either of m or n is equal to 1. Adapt your example to show that part (a) does not necessarily hold if either of m or n is equal to 1.

10. [*May be submitted for marking.*] Let G be an abelian group, let $\pi : \mathbb{Z}^d \to G$ be a homomorphism, and let $B \subset \mathbb{R}^d$ be a symmetric convex body. Show that the set $\pi(B \cap \mathbb{Z}^d)$ is a K-approximate group for some K depending only on d. Noting that a progression is a special case of such a set in which B is a cuboid, formulate a similar generalisation of a Bohr set of rank d, and prove that it is a K-approximate group with K depending only on d.

11. Let $B(\gamma, \rho)$ be a Bohr set of rank d inside a finite abelian group G.
    (a) Show that $B(\gamma, \rho)$ is a 4^d-approximate group.
    (b) A result from the lectures implies that $|B(\gamma, \rho)| \geq (\rho/d)^d |G|$. Prove directly that in fact $|B(\gamma, \rho)| \geq \rho^d |G|$.