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v2

INTRODUCTION TO APPROXIMATE GROUPS—EXAMPLES 2

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1. Show that a coset progression of rank r in an arbitrary abelian group is a Freiman-homomorphic image of a Bohr set of rank r in some finite abelian group.

2. Let Λ be a lattice in $\mathbb{R}^d$ with basis $x_1, \ldots, x_d$, and Γ a sublattice with basis $y_1, \ldots, y_d$. Show that
$$\frac{|\det(y_1, \ldots, y_d)|}{|\det(x_1, \ldots, x_d)|} = [\Lambda : \Gamma].$$

3. Give an example, for some d, of a centrally symmetric convex body $B \subset \mathbb{R}^d$ and a lattice $\Lambda \subset \mathbb{R}^d$ such that whenever $v_1, \ldots, v_d$ is a directional basis for Λ with respect to B we have $\langle v_1, \ldots, v_d \rangle \neq \Lambda$. What is the smallest d for which this is possible?

4. [*May be submitted.*] Set
$$x_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad x_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
let $L_1, L_2 \in \mathbb{N}$, and define $\overline{P}(x_1, x_2; L_1, L_2) = \{x_1^{\ell_1} x_2^{\ell_2} [x_2, x_1]^{\ell_3} : |\ell_1| \leq L_1, |\ell_2| \leq L_2, |\ell_3| \leq L_1 L_2\}$.
   (a) Show that $P_{\mathrm{nil}}(x_1, x_2; L_1, L_2) \subset \overline{P}(x_1, x_2; L_1, L_2) \subset P_{\mathrm{nil}}(x_1, x_2; 5L_1, 5L_2)$.
   (b) Show that the nilprogression $P = P_{\mathrm{nil}}(x_1, x_2; L_1, L_2)$ satisfies $|P^3| \leq K|P|$ for some constant K independent of L_1 and L_2.

5. Give examples to show that a nilprogression $P_{\mathrm{nil}}(x_1, \ldots, x_r; L_1, \ldots, L_r)$ of step s does not necessarily have small doubling if one of the L_i is bounded above in terms of r and s.

6. Let $x_1, \ldots, x_r$ be elements of a group G, and let $L_1, \ldots, L_r \geq 0$. Suppose that there exists some C such that whenever $1 \leq i < j \leq r$ we have
$$[x_i^{\pm 1}, x_j^{\pm 1}] \in P_{\mathrm{ord}} \left(x_{j+1}, \ldots, x_r; \frac{CL_{j+1}}{L_i L_j}, \ldots, \frac{CL_r}{L_i L_j} \right).$$
Show that $\langle x_1, \ldots, x_r \rangle$ is nilpotent of step at most $r - 1$ and that $|P_{\mathrm{ord}}(x; L)^3| \leq f(C, r, s)|P_{\mathrm{ord}}(x; L)|$.

7. Let A be a subset of a group G, let H be another group, and suppose that $\varphi : A \to H$ is a Freiman 2-homomorphism. Show that if G is abelian then so is $\langle \varphi(A) \rangle$. What can you say about $\langle \varphi(A) \rangle$ if G is s-step nilpotent?

8. [*May be submitted.*] In this question the *rank* of a group is the minimum size of a generating set.
   (a) Let G be a finite p-group (you may assume the well-known fact that such groups are nilpotent). Show that all minimal generating sets for G have the same size. Does the same statement hold if G is an arbitrary finite nilpotent group?
   (b) Let G be a finitely generated abelian group, and let H be a subgroup of G. Show that the rank of H is at most the rank of G.
   (c) Let G be a finitely generated nilpotent group of rank r and step s. Show that if H is a subgroup of G then the rank of H is at most $f(r, s)$. Part (b) shows that we may take $f(r, 1) = r$; are there any other values of s for which we may take $f(r, s) = r$?

9. Let G be a group and suppose that $x, y \in G$ generate a non-abelian free subgroup. Write $X = \{x^{\pm 1}, y^{\pm 1}\}$. Show that for every finite set A we have $|AX| \geq 3|A|$.