Lent 2019
v3

INTRODUCTION TO APPROXIMATE GROUPS—EXAMPLES 3

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1. [*May be submitted for marking.*] A classical result of Mal'cev states that if G is a soluble subgroup of $GL_n(\mathbb{C})$ then G contains a subgroup of index $O_n(1)$ that is conjugate to a subgroup of $\mathrm{Upp}_n(\mathbb{C})$.
   (a) Let $A \subset GL_n(\mathbb{C})$ be a finite K-approximate group that generates a soluble subgroup of $GL_n(\mathbb{C})$. Using Mal'cev's result, show that there is some nilpotent subgroup N of step at most n in G such that A is contained in a union of at most $K^{O_n(1)}$ left-cosets of N.
   (b) Deduce that there is a finite subgroup H and a nilprogression P_{nil} of rank at most $K^{O_n(1)}$ and step at most n such that A is contained in the union of at most $K^{O_n(1)}$ left-translates of HP_{nil}, and $|HP_{\text{nil}}| \le \exp(K^{O_n(1)})|A|$.

2. [*May be submitted for marking.*] Show that for each $n \in \mathbb{N}$ there exists a constant $c = c_n > 0$ such that if G is a soluble subgroup of $GL_n(\mathbb{C})$, and if S is a finite symmetric generating set for G containing the identity and satisfying $|S^m| \le m^{c \log m}|S|$ for some $m \ge 2$, then G is virtually n-step nilpotent.

3. Show that a finitely generated infinite group has at least linear growth and at most exponential growth.

4. Show that a finitely generated nilpotent group has polynomial growth.

5. Let F be the free group on generators x, y, and let $S = \{1, x, x^{-1}, y, y^{-1}\}$. Show that $|S^n| = 2 \cdot 3^n - 1$ for every $n \in \mathbb{N}$.

6. Given a group G and $m \in \mathbb{N}$, define $G^m = \langle g^m : g \in G \rangle$ to be the subgroup of G generated by the mth powers of elements of G.
   (a) If G is nilpotent of step s, show that $(G_s)^{m^s} \subset (G^m)_s \subset (G_s)^m$.
   (b) If G is nilpotent of step s and has rank at most r, show that G^m has index at most $m^{O_{r,s}(1)}$ in G.

7. Suppose that G is a finitely generated group of rank at most r, and that $H \lhd G$ is a normal subgroup of size at most k such that G/H is s-step nilpotent. Show that G has an s-step nilpotent subgroup of index at most $O_{r,k,s}(1)$. *This partially refines a lemma from the lectures showing that G has an $(s+1)$-step nilpotent subgroup of index at most $k!$.*

8. If F is the free group on generators $u_1, \dots, u_r$, we define the *free s-step nilpotent group of rank r*, denoted $N_{r,s}$, to be the quotient F/F_{s+1}. More precisely, writing $x_i = u_iF_{s+1} \in N_{r,s}$ for each i, we say that $N_{r,s}$ is the *free s-step nilpotent group on generators $x_1, \dots, x_r$*. Show that an arbitrary s-step nilpotent group of rank at most r is isomorphic to a quotient of $N_{r,s}$ by some normal subgroup.

9. Let G be a group with an s-step nilpotent subgroup N of index at most k, and suppose that S is a finite symmetric generating set for G containing the identity with $|S| \le r$. Show that $|S^n| \le n^{O_{k,r,s}(1)}$ for every $n \ge 2$. *Hint: Show this first when G is the free s-step nilpotent group on generators $x_1, \dots, x_r$ and $S = \{1, x_1^{\pm 1}, \dots, x_r^{\pm 1}\}$. Then treat the case $k = 1$.*

10. A subgroup H of a group G is said to be *characteristic* if $\varphi(H) \subset H$ for every automorphism φ of G.
   (a) Show that characteristic subgroups of a group G are normal in G.
   (b) Show that if G is a group then all the terms of the upper and lower central series are characteristic.
   (c) Let G be a group with subgroups $H < C \lhd G$. Show that if H is characteristic in C then H is normal in G.

(d) Show that if G is a group of rank r and $k \in \mathbb{N}$ then there are at most $O_{r,k}(1)$ subgroups of G of index k. *Hint: If H is a subgroup of index k then there is an action of G on the set of left-cosets of H, which has size k. How many different actions can G have on a set of size k?*

(e) Show that if G is a group of rank r and H is a subgroup of index k then there exists a subgroup $N < H$ of index $O_{r,k}(1)$ that is characteristic in G.

11. Given $d > 0$, show that there exists $N = N_d$ such that if G is a group generated by a finite symmetric set S containing the identity, and if $|S^n| \leq n^d|S|$ for some $n \geq N$, then $|S^m| \leq m^{O_d(1)}|S|$ for every $m \geq n$. *Hints: If C is a group and $H \lhd C$ is such that the quotient C/H is s-step nilpotent, find a subgroup $H' < H$ that is characteristic in C and such that C/H' is s-step nilpotent. Then show that if C is in turn a finite-index normal subgroup of some group G then G/H' is well defined and virtually s-step nilpotent. It may then help to use question 9.*