Nilpotent additive combinatorics (L16)

Graduate Course

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The field of additive combinatorics has been subject to a large volume of research in recent years, both because of its inherent interest and because of its many applications to other fields such as geometric group theory, number theory and differential geometry.

Additive combinatorics is broadly concerned with how finite subsets of groups behave when we add or multiply them together, and in particular how the algebraic structure of those sets governs this behaviour. For example, if A is a random set of size n inside the interval $\{1, 2, \dots, n^{10}\}$ then we expect the set $A+A = \{a+a' : a, a' \in A\}$ to have size roughly quadratic in $|A|$. On the other hand, if A is highly structured then $|A + A|$ can be much smaller than this; for example, if A is a finite arithmetic progression then it is easy to see that $|A + A| \leq 2|A|$.

A classical theorem of Freiman asserts that an arithmetic progression exhibits precisely the kind of structure that prevents $|A + A|$ from being much larger than A. More explicitly, Freiman's theorem states that if $K > 1$ and if $A \subset \mathbf{Z}$ satisfies $|A + A| \leq K|A|$ then A must essentially be a sum of at most $f(K)$ arithmetic progressions (where f is some explicit function).

In the past few years there has been considerable effort to generalise Freiman's theorem to groups other than the integers. The overall aim of this course is to present some of the results and techniques that have emerged from these generalisations. In particular, I will aim to develop the general machinery of so-called *approximate groups* in a way that will equip people attending the course to attack problems that are beyond the scope of these sixteen lectures.

The main result of the course will be a generalisation of Freiman's theorem to nilpotent groups, where it holds with a statement strongly analogous to that in the integers. Once one leaves the nilpotent setting new phenomena begin to emerge, and I will illustrate these phenomena by extending Freiman's theorem further, to residually nilpotent groups. I will also devote some time to applications in geometric group theory. In particular, I will prove (a strengthening of) Gromov's polynomial-growth theorem in the setting of residually nilpotent groups.

Pre-requisites

This course can in a sense be viewed as a sequel to the course *Topics in additive combinatorics*, and for a well-rounded view of the subject it would certainly be preferable to have attended that course first. Nonetheless, this course will be sufficiently self contained that not having attended *Topics in additive combinatorics* should not prevent anyone from understanding the material.

Literature

The course will be essentially self contained, but the following surveys give a high-level overview of the field and some of its applications.

1. E. Breuillard, *Lectures on approximate groups and Hilbert's 5th problem*, Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications **159** (2016), 369-404. Also available at http://arxiv.org/abs/1512.01369.

2. B. J. Green, *Approximate groups and their applications: work of Bourgain, Gamburd, Helfgott and Sarnak*, Current events bulletin of the AMS (2010). Also available at http://arxiv.org/abs/0911.3354.