INTRODUCTION TO APPROXIMATE GROUPS—LIST OF RESULTS

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1. Introduction

Theorem 1.1 (Freiman)**.** *Let G be a group and let $A \subset G$ be a finite subset such that $|A^2| < \frac{3}{2}|A|$. Then there exists a subgroup H with $|H| = |A^2|$ such that for every $a \in A$ we have $A \subset aH = Ha$.*

Lemma 1.2. *Let G be a group and let $A \subset G$ be a finite subset such that $|A^2| < \frac{3}{2}|A|$. Then $H = A^{-1}A$ is a subgroup of G. Moreover, $H = AA^{-1}$ and $|H| < 2|A|$.*

Lemma 1.3. *Let G be a group and let $A \subset G$ be a finite subset such that $|A^2| < \frac{3}{2}|A|$. Set $H = A^{-1}A$ and let $a \in A$. Then $A^2 = aHa$. In particular, $|H| = |A^2|$.*

2. Covering and higher sum and product sets

Theorem 2.1 (Ruzsa)**.** *Let A be a finite subset of the vector space $\mathbb{F}_p^r$, and suppose that $|A + A| \le K|A|$. Then there exists a subspace H of $\mathbb{F}_p^r$ of cardinality at most $p^{K^4}K^2|A|$ such that $A \subset H$.*

Proposition 2.2. *Let A be a finite subset of the vector space $\mathbb{F}_p^r$, and suppose that $|2A - 2A| \le K|A|$. Then there exists a subspace H of $\mathbb{F}_p^r$ of cardinality at most $p^K|A - A|$ such that $A \subset H$. In particular, $|H| \le p^K K|A|$.*

Lemma 2.3 (Ruzsa's covering lemma)**.** *Let A and B be finite subsets of some group and suppose that $|AB|/|B| \le K$. Then there exists a subset $X \subset A$ with $|X| \le K$ such that $A \subset XBB^{-1}$. Indeed, these properties are satisfied by taking X to be a subset of A that is maximal with respect to the property that the translates xB with $x \in X$ are all disjoint.*

Lemma 2.4. *Let A be a finite subset of a group and suppose that $|A^{-1}A^2A^{-1}| \le K|A|$. Then there exists $X \subset A^{-1}A^2$ with $|X| \le K$ such that $A^{-1}A^n \subset X^{n-1}A^{-1}A$ for every $n \in \mathbb{N}$.*

Theorem 2.5 (Plünnecke–Ruzsa)**.** *Let G be an abelian group, and let A be a finite subset of G. Suppose that $|A + A| \le K|A|$. Then $|mA - nA| \le K^{m+n}|A|$ for all non-negative integers m, n.*

Example 2.6. Let H be a finite group, and let $G = H * \langle x \rangle$, the free product of H and the infinite cyclic group with generator x. Set $A = H \cup \{x\}$. Then $|A^2| \le 3|A|$, but A^3 contains HxH, which has size $|H|^2$.

Proposition 2.7. *Let $m \ge 3$ and let $\epsilon_1, \dots, \epsilon_m \in \{\pm 1\}$. Suppose that A is a subset of a group satisfying $|A^3| \le K|A|$. Then $|A^{\epsilon_1} \cdots A^{\epsilon_m}| \le K^{3(m-2)}|A|$.*

Lemma 2.8 (Ruzsa triangle inequality)**.** *Let U, V, W be subsets of a group. Then there exists an injection $\varphi : U \times V^{-1}W \to UV \times UW$. In particular, if U, V, W are finite then $|U||V^{-1}W| \le |UV||UW|$.*

3. Approximate groups

Lemma 3.1. *Let A be a finite approximate group. Then $|A^m| \le K^{m-1}|A|$ for every $m \in \mathbb{N}$.*

Proposition 3.2. *Let A be a finite subset of a group G. If A is a K-approximate group then $|A^3| \le K^2|A|$. Conversely, if $|A^3| \le K|A|$ then there exists an $O(K^{12})$-approximate group B such that $A \subset B$ and $|B| \le 7K^3|A|$. Indeed, we may take $B = (A \cup \{1\} \cup A^{-1})^2$.*

Theorem 3.3. *Let A be a finite subset of a group G satisfying $|A^2| \le K|A|$. Then there exists $U \subset A$ with $|U| \ge \frac{1}{K}|A|$ satisfying $|U^m| \le K^{m-1}|U|$ for every $m \in \mathbb{N}$.*

Lemma 3.4 (Petridis)**.** *Let A and B be finite subsets a group G, and let $U \subset A$ be a non-empty subset of A that minimises the ratio $|UB|/|U|$. Then, writing $R = |UB|/|U|$, for every finite subset $C \subset G$ we have $|CUB| \leq R|CU|$.*

***Theorem 3.5** (Tao; Petridis)**.** *Let A be a finite subset of a group and suppose that $|A^2| \leq K|A|$ and $|AxA| \leq K|A|$ for every $x \in A$. Then $|A^m| \leq K^{O(m)}|A|$ for every $m \geq 3$.*

4. Stability of approximate closure under basic operations

Proposition 4.1 (stability of small tripling under quotients)**.** *Let G, Γ be groups, let $\pi : G \to \Gamma$ be a homomorphism, and let $A \subset G$ be a finite symmetric set containing the identity. Then*

$$\frac{|\pi(A)^m|}{|\pi(A)|} \leq \frac{|A^{m+2}|}{|A|}.$$

In particular, if $|A^3| \leq K|A|$ then $|\pi(A)^3| \leq K^9|\pi(A)|$ by Proposition 2.7.

Lemma 4.2. *Let G be a group, let $H < G$, let $A \subset G$ be a finite set, and let $x \in G$. Then $|A^{-1}A \cap H| \geq |A \cap xH|$.*

Lemma 4.3. *Let G be a group, let $H < G$, write $\pi : G \to G/H$ for the quotient map, and let $A \subset G$ be a finite set. Then $|A^{-1}A \cap H| \geq |A|/|\pi(A)|$.*

Lemma 4.4. *Let G be a group, let $H < G$, write $\pi : G \to G/H$ for the quotient map, and let $A \subset G$ be a finite set. Then $|\pi(A^m)||A^n \cap H| \leq |A^{m+n}|$ for every $m, n \geq 0$.*

Proposition 4.5 (stability of small tripling under intersections with subgroups)**.** *Let G be a group, let $H < G$, and let $A \subset G$ be a finite symmetric set containing the identity. Then*

$$\frac{|A^m \cap H|}{|A^2 \cap H|} \leq \frac{|A^{m+1}|}{|A|}$$

for every $m \in \mathbb{N}$. In particular, if $|A^3| \leq K|A|$ then $|(A^m \cap H)^3| \leq K^{9m}|A^m \cap H|$ for every $m \geq 2$.

Proposition 4.6 (stability of approximate groups under intersections with subgroups)**.** *Let G be a group, let $H < G$, and let $A \subset G$ be a K-approximate group. Then for every $m \in \mathbb{N}$ the intersection $A^m \cap H$ is covered by at most K^{m-1} left translates of $A^2 \cap H$. In particular, $A^m \cap H$ is a K^{2m-1}-approximate group for every $m \geq 2$.*

Lemma 4.7 (stability of small tripling under Freiman homomorphisms)**.** *Let G, Γ be groups, let $A \subset G$ be finite, and suppose that $\varphi : A \to \Gamma$ is a Freiman m-homomorphism. Then $|\varphi(A)^m| \leq |A^m|$. In particular, if φ is injective then*

$$\frac{|\varphi(A)^m|}{|\varphi(A)|} \leq \frac{|A^m|}{|A|},$$

and if φ is a Freiman m-isomorphism then

$$\frac{|\varphi(A)^m|}{|\varphi(A)|} = \frac{|A^m|}{|A|}.$$

Lemma 4.8 (stability of approximate groups under Freiman homomorphisms)**.** *Let G, Γ be groups, let $A \subset G$ be a K-approximate group, and suppose that $\varphi : A^3 \to \Gamma$ is a centred Freiman 2-homomorphism. Then $\varphi(A)$ is also a K-approximate group.*

5. Coset progressions, Bohr sets and the Freiman–Green–Ruzsa theorem

Theorem 5.1 (Freiman for $G = \mathbb{Z}$; Green–Ruzsa for arbitrary G)**.** *Let G be an abelian group, and suppose that $A \subset G$ is a finite subset satisfying $|A + A| \leq K|A|$. Then there exists a coset progression $H + P$ of rank at most $O(K^{O(1)})$ such that $A \subset H + P \subset O(K^{O(1)})(A \cup \{0\} \cup -A)$.*

Proposition 5.2 (partially proved in 'Introduction to Discrete Analysis')**.** *Let G be an abelian group, and suppose that $A \subset G$ is a finite subset satisfying $|A+A| \leq K|A|$. Then there is a subset $B \subset 2A-2A$, a finite abelian group Z satisfying $|Z| \geq |A|$, a set $\Gamma \subset \widehat{Z}$ of size at most $O(K^{O(1)})$, some $\rho \geq 1/O(K^{O(1)})$, and a centred Freiman 2-isomorphism $\varphi : B(\Gamma, \rho) \to B$.*

Proposition 5.3. *Let Z be a finite abelian group, let $\Gamma \subset \widehat{Z}$ with $|\Gamma| = r$, and let $\rho < \frac{1}{2}$. Then there exists a coset progression $H + P \subset B(\Gamma, \rho)$ of rank r with $|H + P| \geq (\rho/r)^r|Z|$.*

Lemma 5.4. *Let $H + P$ be a coset progression of rank r, let G be an abelian group, and suppose that $\varphi : H + P \to G$ is a centred Freiman 2-homomorphism. Then $\varphi(H + P)$ is also a coset progression of rank r.*

6. Geometry of numbers

Lemma 6.1. *Let G be a finite abelian group and let $\Gamma \subset \widehat{G}$. Enumerate $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$, and define $\gamma : G \to \mathbb{R}^d/\mathbb{Z}^d$ via $\gamma = (\gamma_1, \ldots, \gamma_d)$. Then $\gamma(G) + \mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$ with determinant $|\ker \gamma|/|G|$.*

Theorem 6.2 (Minkowski's second theorem)**.** *Let $B \subset \mathbb{R}^d$ be a symmetric convex polytope and let Λ be a lattice. Write $\lambda_1 \leq \ldots \leq \lambda_d$ for the successive minima of B with respect to Λ. Then $\lambda_1 \ldots \lambda_d \operatorname{vol}(B) \leq 2^d \det(\Lambda)$.*

Lemma 6.3 (Blichfeldt)**.** *Let Λ be a lattice in $\mathbb{R}^d$, and let $A \subset \mathbb{R}^d$ be a measurable set. Suppose that A contains no pair of distinct points a, b with $a - b \in \Lambda$. Then $\operatorname{vol}(A) \leq \det(\Lambda)$.*

7. Progressions in the Heisenberg group

No theorems, only examples and definitions.

8. Nilpotent groups

Lemma 8.1. *Let G be a group, let $N, H_1, \ldots, H_k \lhd G$ be normal subgroups of G, and for each i let S_i be a generating set for H_i. Suppose that $[s_1, \ldots, s_k] \in N$ whenever $s_i \in S_i$. Then $[H_1, \ldots, H_k] \subset N$.*

Proposition 8.2. *If $G = G_1 > G_2 > \cdots$ is the lower central series of a group G then $G_{k+1} = [G_k, G]$ for every k. In particular, $[G, \ldots, G]_k = G_k$ for every k.*

Proposition 8.3. *Let G be a group with generating set S. Then $G_k = \langle [s_1, \ldots, s_k]G_{k+1} : s_i \in S\rangle$.*

Proposition 8.4. *Let G be a group, and let $G = G_1 > G_2 > \cdots$ be the lower central series of G. Then $[G_i, G_j] \subset G_{i+j}$ for every $i, j \in \mathbb{N}$.*

Proposition 8.5. *Let G be a nilpotent group, and suppose that $G = H_1 > \cdots > H_{r+1} = \{1\}$ is a central series for G. Then $H_i \supset G_i$ for every $i = 1, \ldots, r + 1$ and $H_{r+1-j} \subset Z_j(G)$ for every $j = 0, \ldots, r$.*

Corollary 8.6. *If a group G is nilpotent then both its upper and lower central series have length exactly $s + 1$, in the sense that $G_s \neq G_{s+1} = \{1\}$ and $Z_{s-1}(G) \neq Z_s(G) = G$.*

9. Torsion-free nilpotent approximate groups: an overview

***Proposition 9.1.** *Given $r, s \in \mathbb{N}$ there exists $\lambda_{r,s} > 0$ such that if $x_1, \ldots, x_r$ are elements in an s-step nilpotent group and $L_1, \ldots, L_r \geq \lambda_{r,s}$ then $P_{\text{nil}}(x; L)$ is an $O_{r,s}(1)$-approximate group.*

Theorem 9.2. *Let G be an s-step nilpotent group, and suppose that $A \subset G$ is a finite K-approximate group. Then there exist a subgroup $H \lhd \langle A\rangle$ and a nilprogression P_{nil} of rank at most $K^{O_s(1)}$ such that $A \subset HP_{\text{nil}} \subset A^{K^{O_s(1)}}$. In particular, $|HP_{\text{nil}}| \leq \exp(K^{O_s(1)})|A|$.*

Theorem 9.3. *Let G be a torsion-free s-step nilpotent group, and suppose that $A \subset G$ is a finite K-approximate group. Then there exists an ordered progression P_{ord} of rank at most $K^{O_s(1)}$ such that $A \subset P_{\text{ord}} \subset A^{K^{O_s(1)}}$.*

Proposition 9.4. *Let G be a torsion-free s-step nilpotent group, and suppose that $A \subset G$ is a finite K-approximate group. Then there exist $k \leq K^{O(1)}$ and $K^{O(1)}$-approximate groups $A_1, \dots, A_k \subset A^{O(1)}$ such that $A \subset A_1 \cdots A_k$.*

Proposition 9.5. *Let G be a torsion-free s-step nilpotent group, and suppose that $A \subset G$ is a finite K-approximate group. Then there exist $r \leq K^{O(1)}$ and $K^{O(1)}$-approximate groups $A_0, A_1 \dots, A_r \subset A^{O(1)}$ such that $|A_0 \cdots A_r| \geq \exp(-K^{O_s(1)})|A|$.*

Theorem 9.6 (Green–Ruzsa)**.** *Let G be an abelian group (written multiplicatively), and suppose that $A \subset G$ is a finite K-approximate group. Then there exist $r \leq K^{O(1)}$, a finite subgroup $H < G$, elements $x_1, \dots, x_r \in G$, and $L_1, \dots, L_r \in \mathbb{N}$ such that $HP(x; L) \subset A^4$ and $|HP(x; L)| \geq \exp(-K^{O(1)})|A|$.*

Proposition 9.7. *Let G be a torsion-free s-step nilpotent group, and suppose that $A \subset G$ is a finite K-approximate group. Write $\pi : G \to G/[G, G]$ for the quotient homomorphism, and noting that $G/[G, G]$ is abelian and $\pi(A)$ is a K-approximate group, let $H < G/[G, G]$ and $x_1, \dots, x_r \in G/[G, G]$ be the subgroup and elements given by applying Theorem 9.6 to $\pi(A)$. Then*

$$\left| \left(A^{24} \cap \pi^{-1}(H)\right) \prod_{i=1}^{r} \left(A^{24} \cap \pi^{-1}(\langle x_i \rangle)\right) \right| \geq \frac{|A|}{\exp K^{O(1)}}.$$

Lemma 9.8. *Let G be an s-step nilpotent group, and write $\pi : G \to G/[G, G]$ for the quotient homomorphism. Then*

(i) for every $x \in G/[G, G]$ the group $\pi^{-1}(\langle x \rangle)$ has step at most $s - 1$; and
(ii) if $H < G/[G, G]$ is a finite subgroup and G is torsion-free then $\pi^{-1}(H)$ has step at most $s - 1$.

Lemma 9.9. *Let G be a group. Then for each $k \in \mathbb{N}$ the simple commutator map*

$$\begin{array}{cccc} [\ ,\dots,\ ]_k & : & G^k & \to & G_k \\ & & (x_1, \dots, x_k) & \mapsto & [x_1, \dots, x_k] \end{array}$$

is a homomorphism modulo G_{k+1} in each variable. Moreover, the commutator subgroup $[G, G]$ is contained in the kernel of each of these homomorphisms in each variable.

10. Torsion-free nilpotent approximate groups: the details

Lemma 10.1. *Let G be a group and $N \lhd G$ a normal subgroup, and write $\pi : G \to G/N$ for the quotient homomorphism. Let $A \subset G$ be symmetric, and define $\varphi : \pi(A) \to A$ by choosing each $\varphi(x)$ arbitrarily so that $\pi(\varphi(x)) = x$. Then for every $a \in A$ we have*

$$a \in \left(A^2 \cap N\right) \varphi(\pi(a)),$$

and for every $x, y \in G/N$ with $x, y, xy \in \pi(A)$ we have

$$\varphi(xy) \in \varphi(x)\varphi(y) \left(A^3 \cap N\right).$$

Lemma 10.2. *Let G be a group, let $U, V < G$, and suppose that $[U, V]$ is central in G. Then the commutator map $[\ ,\ ] : U \times V \to [U, V]$ is a homomorphism in each variable.*

11. p-groups

***Proposition 11.1.** *A finite group is nilpotent if and only if it is a direct product of p-groups.*

Proposition 11.2. *Let Γ be an abelian p-group of rank r and suppose that $X \subset \Gamma$ is a union of subgroups of Γ. Then $\langle X \rangle \subset rX$.*

Lemma 11.3. *Let Γ be a finite abelian p-group. Then a subgroup $H \subsetneq \Gamma$ is maximal if and only if $\Gamma/H \cong \mathbb{Z}/p\mathbb{Z}$.*

Lemma 11.4. *Let Γ be a finite abelian p-group. Then a subset $S \subset \Gamma$ generates Γ if and only if $S + (p \cdot \Gamma)$ generates Γ.*

Corollary 11.5. *Let Γ be a finite abelian p-group. Then all minimal generating sets for Γ have the same size.*

Lemma 11.6. *If Γ is an abelian p-group and Γ' is a subgroup of Γ then the rank of Γ' is at most the rank of Γ.*

***Proposition 11.7.** *Let G be an s-step nilpotent group, let $x_1,\dots,x_r \in G$, and let $L_1,\dots,L_r \in \mathbb{N}$. Then*

$$P_{\mathrm{ord}}(x;L) \subset P_{\mathrm{nil}}(x;L) \subset P_{\mathrm{ord}}(x;L)^{O(s)^{s^2} r^s}.$$

Proposition 11.8. *Let G be a finite p-group of step s, and suppose that $A \subset G$ is a finite K-approximate group. Then there exist $r \leq K^{O(1)}$ and $K^{O(1)}$-approximate groups $A_0, A_1 \dots, A_r \subset A^{O(1)}$ such that $|A_0 \cdots A_r| \geq \exp(-K^{O_s(1)})|A|$, and a normal subgroup $N \lhd G$ with $N \subset A^{K^{O_s(1)}}$ such that $[\,A_i,\dots,A_i\,]_s \subset N$ for each i.*

12. Arbitrary approximate groups

Theorem 12.1 (Breuillard–Green–Tao)**.** *Let G be an arbitrary group and suppose that $A \subset G$ is a finite K-approximate group. Then there exist subgroups $H \lhd \Gamma < G$ with $H \subset A^4$ such that Γ/H is nilpotent of step $O_K(1)$ and A is contained in a union of $O_K(1)$ left-cosets of Γ.*

Corollary 12.2. *Let G be an arbitrary group and suppose that $A \subset G$ is a finite K-approximate group. Then there exist subgroups $H \lhd \Gamma < G$ with $H \subset A^4$ such that Γ/H is nilpotent of step $O_K(1)$ and A is contained in a union of $O_K(1)$ left-translates of $A^2 \cap \Gamma$, which is a K^3-approximate group by Proposition 4.6.*

Lemma 12.3. *Let G be a group with a subgroup Γ, and suppose that $A \subset G$ is a finite K-approximate group such that $|A^m \cap \Gamma| \geq c|A|$. Then A is contained in a union of at most K^m/c left-cosets of Γ.*

13. The sum-product phenomenon over $\mathbb{C}$

Theorem 13.1 (Solymosi's sum-product theorem in $\mathbb{C}$)**.** *Let $U,V,W \subset \mathbb{C}$ be finite sets such that $U,W \neq \{0\}$. Then*

$$|U+V||UW| \geq \frac{|U|^{3/2}|V|^{1/2}|W|^{1/2}}{56}.$$

In particular, an arbitrary finite set $A \subset \mathbb{C}$ satisfies

$$\max\{|A+A|, |AA|\} \geq \frac{|A|^{5/4}}{2\sqrt{14}}.$$

Lemma 13.2. *Suppose that $z_1,\dots,z_k \in \mathbb{C}$ and $r_1,\dots,r_k > 0$ are such that $\bigcap_{i=1}^k \overline{D}(z_i,r_i) \neq \varnothing$ and $z_i \notin D(z_j,r_j)$ whenever $i \neq j$. Then $k \leq 7$.*

Lemma 13.3. *Let $U,V,W \subset \mathbb{C}$ be finite sets and suppose that $0 \neq W$ and $|U| \geq 2$. Fix $v \in V$ and $w \in W$, and for each $u \in U$ fix an element $n(u) \in U \setminus \{u\}$ that minimises $|u - n(u)|$ (thus $n(u)$ is the 'nearest neighbour' to u in U). Then*

$$\sum_{u \in U} \big|\big\{x \in U+V : |(u+v)-x| \leq |u-n(u)|\big\}\big| \leq 7|U+V| \tag{13.1}$$

and

$$\sum_{u \in U} \big|\big\{x \in UW : |uw - x| \leq |uw - n(u)w|\big\}\big| \leq 7|UW|. \tag{13.2}$$

14. Complex upper-triangular groups

Theorem 14.1 (Breuillard–Green)**.** *Suppose that $A \subset \mathrm{Upp}_n(\mathbb{C})$ is a finite K-approximate group. Then there is a nilpotent subgroup $N < \mathrm{Upp}_n(\mathbb{C})$ of step at most n such that $|A^{O_n(1)} \cap N| \geq K^{-O_n(1)}|A|$.*

Lemma 14.2. *Let $\pi : G \to H$ be a homomorphism, let $N < H$, and let $A \subset G$ be a finite symmetric set. Suppose that $|\pi(A^k) \cap N| \geq \alpha|\pi(A)|$. Then $|A^{k+2} \cap \pi^{-1}(N)| \geq \alpha|A|$.*

15. Growth of groups

Lemma 15.1. *The notions of 'polynomial growth of degree d' and 'exponential growth' of a finitely generated group do not depend on the choice of generating set.*

Proposition 15.2. *Let G be a group and $H < G$ a finite-index subgroup. Then H is finitely generated if and only if G is. If they are finitely generated then H has polynomial growth of degree d if and only if G does.*

Lemma 15.3. *Let G be a group with a finite symmetric generating set S containing the identity. Suppose $H < G$ has index at least m. Then S^{m-1} contains representatives of at least m left-cosets of H in G.*

Proposition 15.4. *Let G be a group with a finite symmetric generating set S containing the identity. Suppose that $H < G$ has index $k \in \mathbb{N}$ in G. Let $X \subset S^k$ be a complete set of left-coset representatives for H in G (such a set exists by Lemma 15.3. Then $S^{nk} \subset X(H \cap S^{3k})^{n-1}$ for every $n \geq 2$. In particular, $(H \cap S^{3k})$ generates H.*

Theorem 15.5 (Gromov)**.** *If a finitely generated group G has polynomial growth then G is virtually nilpotent.*

Lemma 15.6. *Let G be a group and let $H < G$ have index $k \in \mathbb{N}$ in G. Then there exists a subgroup $N < H$ with $N \lhd G$ such that $[G : N] \leq k^k$.*

16. A refinement of Gromov's theorem

Theorem 16.1 (Breuillard–Green–Tao)**.** *For every $d > 0$ there exists $N = N_d$ such that if S is a finite symmetric generating set containing the identity for a group G, and $|S^n| \leq n^d|S|$ for some $n \geq N$, then G is virtually nilpotent of step at most $O_d(1)$.*

Proposition 16.2. *Let S be a finite symmetric generating set containing the identity for some group G, and suppose that $|S^n| \leq n^d|S|$ for some $n \geq 3^{12}$. Then there exists $m \in \mathbb{N}$ with $n^{1/2} \leq m \leq n$ such that S^m is a $3^{O(d)}$-approximate group.*

Lemma 16.3. *Let S be a finite symmetric generating set containing the identity for some group G, and suppose that $|S^n| \leq n^d|S|$ for some $n \geq 3^{12}$. Then there exists $m \in \mathbb{N}$ with $\lfloor n^{1/2} \rfloor \leq m \leq n^{5/6}$ and some K depending only on d such that $|S^{3m}| \leq K|S^m|$.*

Proposition 16.4. *For every $d > 0$ there exists $N = N_d$ such that if S is a finite symmetric generating set containing the identity for a group G, and $|S^n| \leq n^d|S|$ for some $n \geq N$, then there exist subgroups $H \lhd C < G$ such that $H \subset S^{4n}$, such that C/H is nilpotent of step at most $O_d(1)$, and such that $[G : C] \leq O_d(1)$.*

Lemma 16.5. *Let G be a group and suppose that $H \lhd G$ is a finite normal subgroup such that G/H is s-step nilpotent. Then there exists a subgroup $N < G$ of index at most $|H|!$ that is nilpotent of step at most $s + 1$.*

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